Homework 9, due Friday, July 29th

In this homework you will explore **variation of parameters** to solve second order linear nonhomogenous differential equations, without necessarily having constant coefficients.

An equation of the above type looks like

$$y'' + f_1(x)y' + f_0(x)y = g(x),$$

where f_1, f_0, g are functions of x. The solution is once a **particular solution** plus the complementary solution, $y = y_p + y_c$, but we use a different technique to get y_p.

To solve these we first **need** the two linearly independent solutions y_1, y_2 to the corresponding homogenous equation

$$y'' + f_1(x)y' + f_0(x)y = 0.$$

In other words we need the **complementary solution** $y_c = C_1y_1 + C_2y_2$ before we can hope to find y_p.

We used characteristic equations to get y_c when we had constant coefficients like with,

$$y'' + a_1y' + a_0y = 0.$$

but unfortunately now we have functions as coefficients so we don't have a method for finding y_c and instead have to be given it or just given y_1, y_2.

However, once we know y_1 and y_2 we know that the following formula derived in class will give us a **particular solution**.

$$y_p = -y_1 \int \frac{y_2 g(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)} dx$$

1) For each of the following differential equations and complementary solutions, write down a formula which gives you y_p, but **do not evaluate the integrals.** (Make sure they are in the right form before you start!!! I.e. always divide out anything in front of y'').

a) $y'' + \frac{1}{x}y' - \frac{1}{x^2}y = x^2 \cos(e^{4x}) \ln(x)$ where $y_c = C_1x + C_2\frac{1}{x}$

b) $y'' + 4xy + (4x^2 + 2)y = 6e^{x^2} \ln(\sin(3x))$ where $y_c = C_1e^{-x^2} + C_2xe^{-x^2}$

c) $x^2y'' - 3xy' + 3y = 2^x \cos(\ln(x^{-1})) \sin(3x)$ where $y_c = C_1x + C_2x^3$

These examples show that this method can potentially tackle much more difficult problems than the method of undetermined coefficients did, or at least in theory, but the integrals might be way too hard so we still can't get the particular solution!

Now for some solvable problems! Some of these may require more advanced integration techinques like integration by parts.

2) Find a particular solution y_p of

$$y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y = x - 1$$

given that $y_1 = x$ and $y_2 = e^x$ are solutions of homogeneous version. Then find the general solution.

3) Find a particular solution y_p of

$$x^2y'' - 2xy' + 2y = x^{9/2}$$

given that $y_1 = x$ and $y_2 = x^2$ are solutions of homogeneous version. Then find the general solution.

4) Solve

$$(x^2 - 1)y'' + 4xy' + 2y = \frac{2}{x+1}$$

such that the general solution obeys the initial conditions $y(0) = -1$ and $y'(0) = -5$ and given that $y_1 = \dfrac{1}{x-1}$ and $y_2 = \dfrac{1}{x+1}$ are solutions to the homogeneous version.

As a check to our sanity we should try to do the same problem in two different ways to make sure everything is working correctly.

5) Solve the following differential equation first using the method of undetermined coefficients and then with variation of parameters.

$$y'' + 4y' - 5y = 8x^2$$

Practice Problems - Do not turn these in

Doing the following problems will benefit you. Practice makes perfect and **math is not a spectator sport**.

P1) Solve the following differential equations using variation of parameters where the solutions to the homogeneous version are given. Some integrals may require more advanced techniques to solve, but you should be able to set them all up.

a) $4xy'' + 2y' + y = \sin(\sqrt{x})$ where $y_1 = \cos(\sqrt{x})$ and $y_2 = \sin(\sqrt{x})$

b) $xy'' - (2x + 2)y' + (x + 2)y = 6x^3e^x$ where $y_1 = e^x$ and $y_2 = x^3e^x$

c) $x^2y'' - 4xy' + (x + 6)y = x^4$ where $y_1 = x^2\cos(x)$ and $y_2 = x^2\sin(x)$