Homework 4, due Tuesday July 5th

In this homework you will explore exact equations and integrating factors.

1) Consider the differential equation given by:

$$(x + \sin(y))dx = -(x\cos(y) - 2y)dy$$

.

a) Write the equation in the form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$. What is $M(x,y)$? What is $N(x,y)$?

If we take a multivaribale function $\Psi(x,y)$ and take the multivariable derivatve $\frac{d}{dx}[\Psi(x,y)] = \frac{\partial\Psi}{\partial x} + \frac{\partial\Psi}{\partial y}\frac{dy}{dx}$ we should notice the similarities to the rewritten differential equation. It's easier to see when written like this:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

$$\frac{\partial\Psi}{\partial x} \quad + \quad \frac{\partial\Psi}{\partial y}\ \frac{dy}{dx}$$

This is why for a function $\Psi(x,y)$ to be a solution of the differential equation we need that $\frac{\partial\Psi}{\partial x} = M(x,y)$, $\frac{\partial\Psi}{\partial y} = N(x,y)$, and $\Psi(x,y) = c$. But when is it possible to find such a $\Psi(x,y)$?

We need the fact that **if you take the partial derivative with respect to two variables in either order the result is always equal** i.e.

$$f_{xy} = f_{yx}$$

b) Calculate M_y and N_x.

These should be equal. Remember if $\Psi(x,y)$ is to be a solution then $\Psi_x = M$ and $\Psi_y = N$. Then finding M_y tells us $\Psi_{xy} = M_y$ and finding N_x tells us that $\Psi_{yx} = N_x$. But according to the fact above

$$M_y = \Psi_{xy} = \Psi_{yx} = N_x$$

. This is why testing if $M_y = N_x$ will tell us we will be able to find such a $\Psi(x,y)$. Since it works for our problem lets do it!.

c) Since we need $\Psi_x = M(x,y)$, take the integral of $M(x,y)dx$ while treating y as a constant.

d) Since we need $\Psi_y = N(x,y)$, take the integral of $N(x,y)dy$ while treating x as a constant.

Recall that when you integrated in math 9B you had to add a $+C$ because there could be a hidden constant. When we integrate with respect to x we need to add a $+h(y)$ and when we integrate with respect to y we add a $+k(x)$ because there could be hidden functions of the variable you aren't integrating with respect to.

From c) we know $\Psi(x,y) =$ ---------- $+ h(y)$ for some function $h(y)$.

From d) we know $\Psi(x,y) =$ ---------- $+ k(x)$ for some function $k(x)$.

Together these two pieces of information tell you everything you need to know about $\Psi(x,y)$, namely it needs everything from both parts, but not duplicates!

e) Write your solution as $\Psi(x,y) = c$ to complete the problem.

f) Calculate the multivariable derivative $\frac{d}{dx}[\frac{x^2}{2} + x\sin(y) - y^2]$ using the formula $\frac{d}{dx}[\Psi(x,y)] = \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{dy}{dx}$.

2) Check if the following differential equations are exact. If so then find the solution $\Psi(x,y) = c$.

a) $(xy+1)dx + (xy-1)dy = 0$

b) $3x^2y^2dx + (2x^3y + 4y^3)dy = 0$

c) $e^{xy}dx + \cos(x)dy = 0$

3) Show the differential equation is separable, Bernoulli, and exact. You do not need to solve it.

$$y' = xy^3$$

Some equations aren't exact, but we can multiply by an integrating factor to get a new equation which is exact and has the same solution. Note that **integrating factor** is an umbrella term, it doesn't just mean the things we used to solve linear equations. It means something you multiply to a differential equation to make it better.

4) Consider the not exact differential equation:

$$(y+1)dx - xdy = 0$$

.

a) Show it's not exact.

b) Multiply it by the integrating factor $\frac{1}{x^2}$ and show it is now exact.

c) Solve this new exact differential equation.

Practice Problems - Do not turn these in

Doing the following problems will benefit you. Practice makes perfect and **math is not a spectator sport**.

P1) Test whether the following are exact and solve those that are.

a) $ydx + xdy = 0$

b) $(y\sin(x) + xy\cos(x))dx + (x\sin(x) + 1)dy = 0$

c) $(4t^3y^3 - 2ty)dt + (3t^4y^2 - t^2)dy = 0$

d) $(t^2 - x)dt - tdx = 0.$

e) $2xe^{2t}dt + (1 + e^{2t})dx = 0.$

f) $y^2dt + t^2dy = 0.$