Homework 4, due Tuesday July 5th

In this homework you will explore exact equations and integrating factors.

1) Consider the differential equation given by:

\[(x + \sin(y))dx = -(x \cos(y) - 2y)dy\]

a) Write the equation in the form \(M(x, y) + N(x, y) \frac{dy}{dx} = 0\). What is \(M(x, y)\)? What is \(N(x, y)\)?

If we take a multivariable function \(\Psi(x, y)\) and take the multivariable derivative \(\frac{d}{dx}[\Psi(x, y)] = \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y} \frac{dy}{dx}\) we should notice the similarities to the rewritten differential equation. It’s easier to see when written like this:

\[M(x, y) + N(x, y) \frac{dy}{dx} = 0\]

\[\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y} \frac{dy}{dx}\]

This is why for a function \(\Psi(x, y)\) to be a solution of the differential equation we need that \(\frac{\partial \Psi}{\partial x} = M(x, y), \frac{\partial \Psi}{\partial y} = N(x, y),\) and \(\Psi(x, y) = c\). But when is it possible to find such a \(\Psi(x, y)\)?

We need the fact that if you take the partial derivative with respect to two variables in either order the result is always equal i.e.

\[f_{xy} = f_{yx}\]

b) Calculate \(M_y\) and \(N_x\).

These should be equal. Remember if \(\Psi(x, y)\) is to be a solution then \(\Psi_x = M\) and \(\Psi_y = N\). Then finding \(M_y\) tells us \(\Psi_{xy} = M_y\) and finding \(N_x\) tells us that \(\Psi_{yx} = N_x\). But according to the fact above

\[M_y = \Psi_{xy} = \Psi_{yx} = N_x\]

This is why testing if \(M_y = N_x\) will tell us we will be able to find such a \(\Psi(x, y)\). Since it works for our problem lets do it!.

c) Since we need \(\Psi_x = M(x, y)\), take the integral of \(M(x, y)dx\) while treating \(y\) as a constant.
d) Since we need $\Psi_y = N(x, y)$, take the integral of $N(x, y)dy$ while treating $x$ as a constant.

Recall that when you integrated in math 9B you had to add a $+C$ because there could be a hidden constant. When we integrate with respect to $x$ we need to add a $+h(y)$ and when we integrate with respect to $y$ we add a $+k(x)$ because there could be hidden functions of the variable you aren’t integrating with respect to.

From c) we know $\Psi(x, y) = \_\_\_\_ + h(y)$ for some function $h(y)$.

From d) we know $\Psi(x, y) = \_\_\_\_ + k(x)$ for some function $k(x)$.

Together these two pieces of information tell you everything you need to know about $\Psi(x, y)$, namely it needs everything from both parts, but not duplicates!

e) Write your solution as $\Psi(x, y) = c$ to complete the problem.

f) Calculate the multivariable derivative $\frac{d}{dx}[\frac{x^2}{2} + x\sin(y) - y^2]$ using the formula $\frac{d}{dx}[\Psi(x, y)] = \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y} \frac{dy}{dx}$.

2) Check if the following differential equations are exact. If so then find the solution $\Psi(x, y) = c$.

a) $(xy + 1)dx + (xy - 1)dy = 0$

b) $3x^2y^2dx + (2x^3y + 4y^3)dy = 0$

c) $e^{xy}dx + \cos(x)dy = 0$

3) Show the differential equation is separable, Bernoulli, and exact. You do not need to solve it.

$y' = xy^3$

Some equations aren’t exact, but we can multiply by an integrating factor to get a new equation which is exact and has the same solution. Note that integrating factor is an umbrella term, it doesn’t just mean the things we used to solve linear equations. It means something you multiply to a differential equation to make it better.

4) Consider the not exact differential equation:

$(y + 1)dx - xdy = 0$
a) Show it’s not exact.

b) Multiply it by the integrating factor $\frac{1}{x^2}$ and show it is now exact.

c) Solve this new exact differential equation.
Practice Problems - Do not turn these in

Doing the following problems will benefit you. Practice makes perfect and **math is not a spectator sport.**

P1) Test whether the following are exact and solve those that are.

a) \( ydx + xdy = 0 \)

b) \((y\sin(x) + xy\cos(x))dx + (x\sin(x)+1)dy = 0\)

c) \((4t^3y^3 - 2ty)dt + (3t^4y^2 - t^2)dy = 0\)

d) \((t^2 - x)dt - tdx = 0.\)

e) \(2xe^{2t}dt + (1 + e^{2t})dx = 0.\)

f) \(y^2dt + t^2dy = 0.\)