Homework 10, due Tuesday August 2nd

Recall from class that the motion of an object on a spring is modeled by a second order linear differential equation with constant coefficients:

$$p'' + \frac{b}{m}p' + \frac{k}{m}p = \frac{F(t)}{m}$$

where p stands for $p(t)$, the position of the object at any time t, m is the mass of the object, b is the constant of proportionality for air resistance, k is the spring constant, and $F(t)$ is an external force.

Also a spring can be put in motion by changing the initial position away from equilibrium, giving it an initial velocity, or even both at the same time. These correspond to initial conditions, which as usual are only inserted **after** we've figured out $p(t)$ and that they are used to find the constants in $p(t)$ specific to the problem.

1) A steel ball weighting 64 pounds is suspended from a spring, whereupon the spring is stretched 4 feet from its natural length. The ball is started in motion with no initial velocity by displacing it 15 feet below the equilibrium position. Additionally we assume that there is no air resistance.

a) Hooke's law tells us that whenever a force is applied to a spring, it is proportional to the length stretched, which gives us the spring constant k. In this case when the ball is attached, the force being applied is mg caused by gravity where $g = 32ft/sec^2$. This is the same as the weight of the ball, so -64 pounds is the force and the stretch corresponding to this force is 4 ft. Use this to find k.

b) Since weight equals mg we can find m by doing $64/32$. Now we have all the constants to proceed. Write down the differential equation modeling this problem. (It should be homogeneous.)

c) Solve this differential equation for $p(t)$.

d) Now we use our initial conditions. Since to **start** the motion we hold the ball 15 feet below equilibrium, this means at time $t = 0$, the position is 15, i.e. $p(0) = 15$. Since there is no initial velocity $p'(0) = 0$. Use these conditions to get the $p(t)$ corresponding to this particular problem.

We should attempt to analyze the solutions we find to see if they match our intuition. With no air resistance, friction, or any additional force the spring should continue in motion forever in a periodic manner. This is because there is nothing to stop it from moving or to slow it down.

e) Explain why your solution follows the intuition described in the above paragraph.

If air resistance is present then the motion should eventually come to a stop as the amount of time passing approaches infinity. Let's do a problem with air resistance and see if that happens.

2) A $10kg$ mass is attached to a spring, stretching it $1.225m$ from it's natural length. The mass is started in motion from the equilibrium position with initial velocity of $1m/s$ in the upward direction.

a) Find the motion of the mass at any time t if the force due to air resistance is $-70p'N$.

b) Explain why the solution follows along with the intuition.

Next if an external force is applied then the motion of the spring should inevitably follow along with the motion of the external force. At first the natural motion plays a part, but if air resistance is present we know the motion slows down and eventually will stop. However the external force takes control and forces the spring to do what it wants! So if the external force is sinusoidal for example, the motion of the spring should reflect that **eventually** (as time goes to infinity).

3) A $10kg$ mass is attached to a spring with a spring constant of $80N/m$. The mass is started in motion from the equilibrium position with an initial velocity of $1m/s$ upward and with an applied external force $F(t) = 50\sin(t)$.

a) Find the motion of the mass at any time t if the force due to air resistance is $-70p'N$.

b) Explain why the solution follows along with the intuition.

Practice Problems - Do not turn these in

Doing the following problems will benefit you. Practice makes perfect and **math is not a spectator sport**.

P1) A 10 pound weight is suspended from a spring and it stretches it 2 inches from its natural length. Find the spring constant.

P2) A mass of $.5kg$ is hung onto a spring and stretches it $1m$ from its natural length. Find the spring constant.

P3) A $20kg$ mass is suspended from the end of a vertical spring having spring constant of $40N/m$ and is allowed to reach equilibrium. It is then set into motion by stretching the spring $2m$ from its equilibrium position and releasing the mass from rest. Find the position of the weight at any time t if there is no external force or air resistance.

P4) A $32kg$ mass is attached to a spring stretching it 8 meters from its natural length. The mass is started in motion by displacing it 1 meter in the upward direction and by giving it an initial velocity of $2m/s$ in the downward direction. Find the motion of the mass if there is no air resistance.

P5) A $.5kg$ mass is attached to a spring having a spring constant $6N/m$ and allowed to come to rest. The mass is set into motion by applying an external force $F(t) = 24\cos(3t) - 33\sin(3t)$. Find the subsequent motion of the mass if the air resistance has a force of $-3p'N$.